FSF April Math Competition Solutions

April 2024

1. Anna, Benjamin, and Chloe are sharing a pizza. Anna takes $\frac{1}{5}$ of the pizza, Benjamin takes $\frac{1}{3}$ of the remaining pizza, and Chloe takes some too. After all three take their pizza, there is still $\frac{1}{15}$ of the pizza remaining. What fraction of the whole pizza did Chloe take?

   Solution. The problem begins with one whole pizza. After Anna takes $\frac{1}{5}$ of the pizza, there is still $1 - \frac{1}{5}$ or $\frac{4}{5}$ of the pizza left. Then Benjamin takes $\frac{1}{3}$ of the remaining $\frac{4}{5}$ of the pizza, or $\frac{4}{5} \cdot \frac{1}{3} = \frac{4}{15}$ of the pizza. This results in $\frac{4}{5} - \frac{4}{15} = \frac{8}{15}$ of the pizza remaining. Finally, Chloe must take away $\frac{8}{15} - \frac{1}{15} = \frac{7}{15}$ of the pizza for there to be $\frac{1}{15}$ of the pizza remaining. Therefore, the answer to this problem is $\frac{7}{15} \rightarrow \mathbf{B}$.

2. Leo has an urn of balls. When he takes them out in groups of three, he has two balls remaining. When he takes them out in groups of five, he has three balls remaining. When he takes them out in groups of seven, he has four balls remaining. If he has less than 100 balls, how many balls are in his urn?

   Solution. The four answer choices can be analyzed in this problem. 33 and 63 cannot be the correct answers because if Leo takes them out in groups of three, he will have zero balls remaining instead of two. 43 also cannot be the correct answer, because only one ball remains when taken out in groups of three. That leaves us with 53, which fulfills all of the conditions above. Therefore, the answer to this problem is $53 \rightarrow \mathbf{C}$.

3. How many ways are there to arrange the letters of the phrase "I love math" ignoring spaces?

   Solution. Since all the letters in the phrase "I love math" are distinct, and there are 9 letters total, the number of ways to arrange the letters would be 9!, or $362880 \rightarrow \mathbf{A}$.

4. Let x be the answer to this question. If the probability of raining today is x and the probability of raining tomorrow is independent of today with probability $2x$, what is the probability that it will not rain today and rain tomorrow?

Solution. The probability that it will not rain today is $1 - x$, and the probability it will rain tomorrow is $2x$, so the probability that it will not rain today and rain tomorrow would be the product of these two values, $2x(1-x)$, or $2x-2x^2$. Since the answer to this question is x, we know that $2x - 2x^2 = x$. Therefore, $2x^2 = x$, and x must either be $\frac{1}{2}$ or 0. Looking at the answer choices, $\frac{1}{2}$ is an answer when 0 is not, so the answer to this problem is $\frac{1}{2} \rightarrow$ **B**.

5. There are 40 pairs of socks in a drawer. 10 of these pairs contain 2 red socks, 20 pairs contain one red sock and one blue sock, and 10 pairs contain 1 blue sock and 1 green sock. I draw a pair of socks at random, I see 1 blue sock, what is the probability that the other sock is also blue?

   Solution. If the other sock is blue, that means that the pair of socks consists of 2 blue socks. There are no pairs of 2 blue socks in the drawer, making the probability $0 \rightarrow$ **A**.

6. 24 students are in a class learning how to make cookies. The recipe asks them to put butter and sugar into the bowl. All the boys and one sixth of the girls put in butter, and all of the girls and one sixth of the boys put in sugar. How many students have a missing ingredient?

   Solution. Since all the boys put in butter but only 1/6 of the boys put in sugar, 5/6 of the boys have a missing ingredient. Since all the girls put in sugar but only 1/6 of the girls put in butter, 5/6 of the girls have a missing ingredient. Therefore, adding the boys and girls together, 5/6 of the total population have a missing ingredient, and the answer to the problem is $24 \cdot \frac{5}{6}$, or $20 \rightarrow$ **D**.

7. In a magic square, the sum of every row, column, and diagonal is the same. In a 3 by 3 square, if the top left square is 5, the center square is 4, and the bottom right square is 10, what is the sum of the numbers in all 9 squares?

   Solution. Since the top left square, center square, and bottom right square make up a diagonal, their sum is equal to the sum of every row. This means that the sum of the values in each row is $5 + 4 + 10 = 19$, and since there are 3 rows total, the sum of the numbers in all 9 squares would be $3 \cdot 19 = 57 \rightarrow$ **D**.

8. What is the sum of the first 20 terms of an arithmetic sequence with 2nd term 6 and 9th term 55?

   Solution. The common difference of this arithmetic sequence can be found by finding the difference of 55 and 6 and dividing it by $9 - 2 = 7$. Therefore, we know that the common difference is 7. The 20th term of this sequence can be found by adding $20-2 = 18$ times the common difference to the 2nd term. Therefore, the 20th term is $6 + 18 \cdot 7 = 132$. The first term can be found similarly to get -1. The sum of the first n terms in an arithmetic sequence can be modeled by $\frac{n(a_1+a_n)}{2}$ where a_i is the ith term of the sequence. Plugging in values, we get that the sum of the first 20 terms of the sequence is $\frac{20(-1+132)}{2} = 1310 \to$ **B**.

9. Three unit fractions sum to 1. However, when you take their individual reciprocals and add them up again, the sum becomes 10. What is the denominator of the product of these three fractions?

   Solution. Since the sum of the denominators is 10, and they are all positive integers, not many possibilities are left, as the denominators cannot be large numbers. Through some guess and check, it is easy to find that $\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$ and $2 + 4 + 4 = 10$. Therefore, $\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{32}$, and the denominator of $\frac{1}{32}$ is $32 \to$ **C**.

10. A solid shape A has a surface area 12 and volume 24. It is then diluted so that the new solid, A', has a surface area of 48. What is the volume of A'?

    Solution. Surface area is measured in units^2, meaning that when it is diluted, the ratio of the surface areas is the square of the dilution factor. Since the ratio of the new surface area to the old surface area is $48/12 = 4$, the dilution factor is $\sqrt{4} = 2$. Volume is measured in units^3, so the ratio of the volumes is the cube of the dilution factor. Since the dilution factor is 2, the volume of A' is 2^3 or 8 times the volume of A. Therefore, the volume of A' is $24 \cdot 8 = 192 \to$ **D**.

11. Find the sum of all the factors of 2024.

    Solution. The prime factorization of 2024 is $2^3 \cdot 11 \cdot 23$, so the sum of all the factors is $(1 + 2 + 4 + 8)(1 + 11)(1 + 23) = 4320 \to$ **A**.

12. Let a, b, c be the roots to $x^3 - 6x^2 + 4x - 8$. Find $a^3 + b^3 + c^3$.

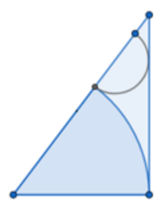
    Solution. $a^3+b^3+c^3$ can be written as $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)+3abc$. $a^2+b^2+c^2$ can also be written as $(a+b+c)^2-2(ab+bc+ca)$, resulting in $a^3+b^3+c^3 = (a+b+c)((a+b+c)^2-3(ab+bc+ca))+3abc$. Using Vieta's Formula, $a + b + c = 6$, $ab + bc + ca = 4$, and $abc = 8$. Therefore, $a^3 + b^3 + c^3 = (6)(6^2 - 3(4)) + 3(8) = 168 \to$ **B**.

13. What is

    $$1 \cdot 2 \cdot 3 \cdot 4 + 2 \cdot 3 \cdot 4 \cdot 5 + \cdots + 11 \cdot 12 \cdot 13 \cdot 14?$$

    Solution. Since these are all groups of 4, there must be exactly one multiple of 4 and one multiple of 2 (that isn't a multiple of 4).There also must be at least one multiple of 3 in each group. Therefore, we can factor out 24 from each group of 4, meaning that the final sum must also be a multiple of 24. Looking at the answer choices, B and D are not divisible by 24, so they are definitely wrong. To find which of the two remaining answer choices is correct, we can use divisibility by 5. The only groups not divisible by 5 are $1 \cdot 2 \cdot 3 \cdot 4$, $6 \cdot 7 \cdot 8 \cdot 9$, and $11 \cdot 12 \cdot 13 \cdot 14$. The one's digit of this sum is 2, meaning that the final sum must have a remainder of 2 when divided by 5, and answer A does not fit this condition. Therefore, the correct answer is $72072 \rightarrow$ **C**.

14. In the figure below, a circle sector is drawn within a 3-4-5 right triangle. A semicircle is then drawn from the point where the circle sector meets the hypotenuse and is tangent to the leg of length 4. Find the radius of the semicircle.



Solution. Let A, B, and C be the vertices of the triangle such that $AB = 3$, $BC = 4$, and $CA = 5$.Let D be the center of the semicircle, and let E be the point where the semicircle touches BC. Connect a line from D to E. Since BC is tangent to the semicircle at E, and D is the center of the semicircle, DE is perpendicular to BC and therefore $\triangle AED \sim \triangle ABC$. Using proportions from the similar triangles and letting r be the radius of the semicircle, $\frac{r}{3} = \frac{2-r}{5}$. Solving for r, we get $5r = 6 - 3r$, or $8r = 6$. Therefore, $r = \frac{6}{8} = \frac{3}{4} \rightarrow$ **C**.

15. The sum of the first two terms of a geometric sequence is 16. The product of the first three terms of that sequence is 1728. What is the fourth term?

Solution. Let a be the 1st term in the sequence and r be the ratio between each term. Then, we can write $a + ar = 16$ and $a \cdot ar \cdot ar^2 = (ar)^3 = 1728$. The second equation gives us $ar = 12$ after taking the cube root of both sides. Plugging this into the first equation, we get $a + 12 = 16$ or $a = 4$. Since $ar = 12$ and $a = 4$, $r = ar/a = 3$. The fourth term of the sequence can be expressed as ar^3, so the fourth term is $4(3)^3 = 108 \rightarrow$ **A**.

16. Imagine the Cartesian plane as a big pool table. There is a circular barrier centered at the origin with radius of $\sqrt{2}$. If a person who is at the point $(-\sqrt{2}, 0)$ strikes a pool ball towards the point $(1, 1)$, the ball will bounce off the rim and keep going until it hits the rim again. What are the coordinates of the point where the ball hits the rim the second time?

Solution. The path that the ball will follow after hitting the rim is a reflection of the path it came from over the line connecting the origin of the circle and the point it hit. Since the circle's origin is at $(0, 0)$ and the ball hit the rim at $(1, 1)$, the line the path should be reflected over is $f(x) = x$. The point that it was stroked from, $(-\sqrt{2}, 0)$, is on the rim, so the point that is reflected from $(-\sqrt{2}, 0)$ will be the place the ball strikes the rim a second time. Reflecting $(-\sqrt{2}, 0)$ over $f(x) = x$ yields the point $(0, -\sqrt{2}) \to$ **B**.

17. If a triangle has side lengths 3, 5, and 7, what is the product of the radii of the incircle and circumcircle?

Solution. Let S be the area of the triangle, a, b, and c be the sides of the triangle, R be the radius of the circumcircle, and r be the radius of the incircle.

By Law of Sines, $\frac{a}{\sin\alpha} = 2R$, with α being the angle across from a. Also, $\frac{1}{2}bc \sin\alpha = S$, so if we put these together, we get $\frac{abc}{2S} = \frac{abc}{2 \cdot \frac{1}{2}bc \sin\alpha} = \frac{a}{\sin\alpha} = 2R$. Therefore, $abc = 4SR$.

On the other hand, we can find S in terms of r. If we call the incenter of triangle point D and the 3 vertices A, B, and C, then we can represent S as the sum of the areas of $\triangle ABD$, $\triangle BCD$, and $\triangle CAD$. The areas of these triangles would just be $\frac{1}{2}$ of the side length times r. Therefore, $S = \frac{r(a+b+c)}{2}$, or $a + b + c = \frac{2S}{r}$.

Dividing $a + b + c = \frac{2S}{r}$ from $abc = 4SR$, we get $\frac{abc}{a+b+c} = 2Rr$, or $\frac{abc}{2(a+b+c)} = Rr$. Substituting the side lengths 3, 5, and 7, leads us to $Rr = \frac{3 \cdot 5 \cdot 7}{2(3+5+7)} = \frac{7}{2} \to$ **C**.

18. In a test consisting of two parts, six individuals received scores of 23, 24, 26, 28, 28, and 29 in the first part and 10, 10, 12, 12, 14, and 16 in the second part, though not in the same order. The overall winner is determined by the highest total score from both parts. What is the minimum possible total score the winner could have?

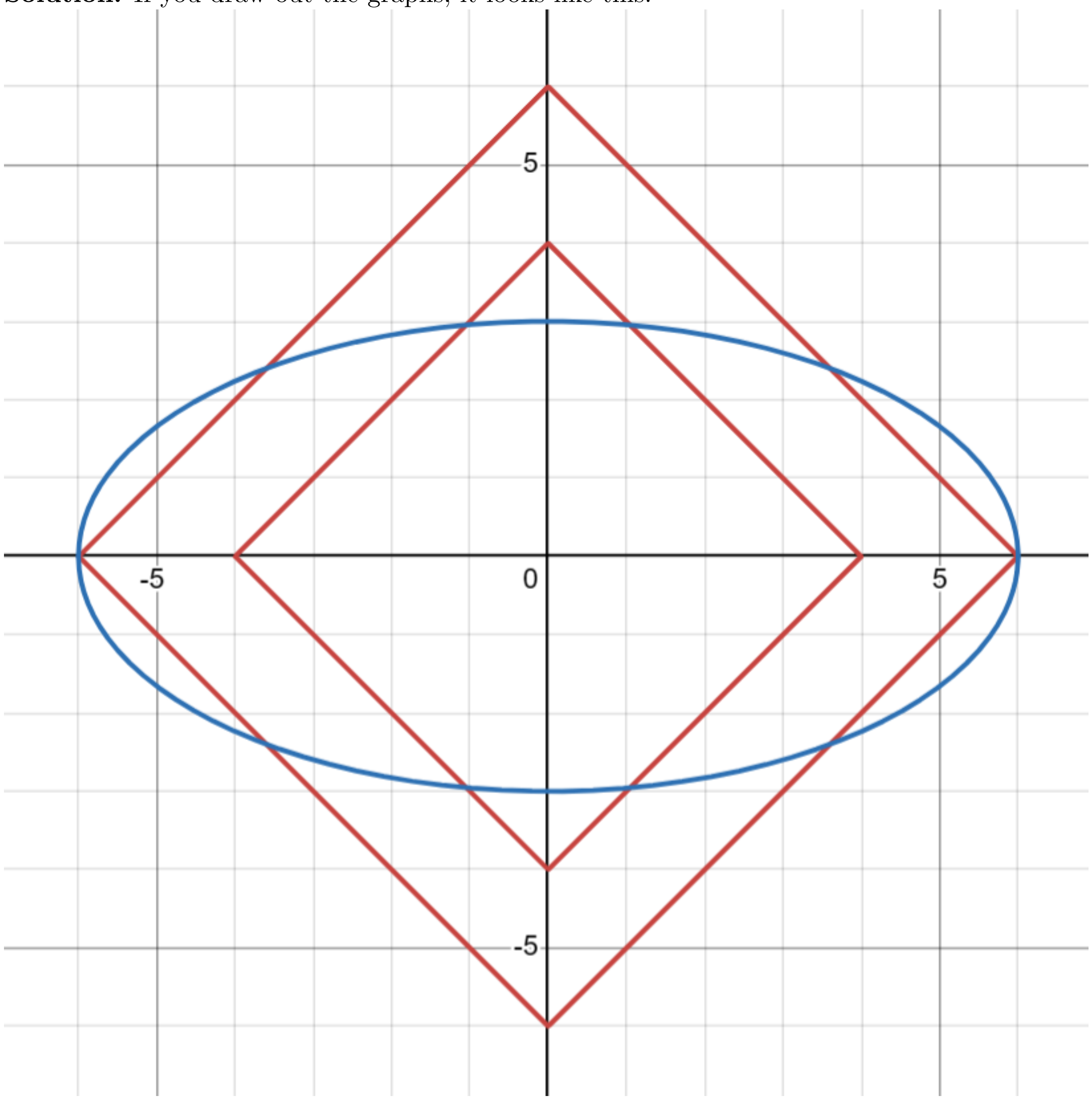
Solution. In order to find the minimum possible winning score, we need to have all the other scores as close to the winning score as possible. The average score (if all individuals received the same score) would be around 38.67 points, so ideally, the winning score should be just a little higher than this average. Looking at the answer choices, 40 is the closest value, so if the 12s and the 28s are matched, and the 24 and 26 are paired with 16 and 14, respectively, then all that is left is $23 + 10$ and $29 + 10$ which are both less than 40. Therefore, the answer would be $40 \to$ **C**.

19. For the set of equations

$$||x| + |y| - 5| = 1$$
$$\frac{x^2}{36} + \frac{y^2}{9} = 1$$

How many real solutions (x, y) are there?

Solution. If you draw out the graphs, it looks like this:



It is evident that the number of intersection points is $10 \to \mathbf{D}$.

20. If $\gcd(a,b) + \text{lcm}(a,b) + 135 = \gcd(a,b) * \text{lcm}(a,b)$, where a and b are positive integers. What is the sum of all possible values of $a+b$?

Solution: Note that $\gcd(a,b) \cdot lcm(a,b) = ab$, so we have

$$\gcd(a,b) + lcm(a,b) + 135 = ab.$$

Let d be the gcd, and $a = dx, b = dy$, so we have

$$d + dxy + 135 = d^2xy \to 135 = d(dxy - xy - 1).$$

Notice that d must be a factor of 135. If $d = 1$, then we get -1 on the right hand side which is impossible. If $d = 3$, we get

$$135 = 3(2xy - 1) \to \ 45 = 2xy - 1 \to xy = 23.$$

So, x, y are $23, 1$ in some order, as 23 is prime, doesn't matter which is which. So, a, b is $3, 69$ in some order, in this case $a + b$ is 72. Similarly, if you check $d = 5$, you get a, b is $35, 5$. And for $d = 9$ you get a, b is $18, 9$. So, the answer is $72 + 27 + 40 = 139$. Answer is B.