FSF Math Competition Middle School Exam

February 2025

1. Alice, Bob, Claire, and David are sitting in a row, with seats numbered 1, 2, 3, and 4. It is known that David is sitting in an even number seat and that Bob and Claire are sitting in prime number seats. What seat is Alice sitting in?

   (a) 1 ← CORRECT

   (b) 2

   (c) 3

   (d) 4

   Solution: Since Bob and Claire are sitting in prime number seats, it is known that they are in seats 2 and 3, since those are the only prime number seats. It is also known that David is sitting in an even number seat, so the only even seat left is seat 4. That leaves Alice in seat $\boxed{1}$.

2. Evaluate $355^2 - 345^2$.

   (a) 3000

   (b) 5000

   (c) 7000 ← CORRECT

   (d) 9000

   Solution: $355^2 - 345^2 = (355 + 345)(355 - 345) = 700 \cdot 10 = \boxed{7000}$.

3. How many of the first 10 even numbers can be expressed as a sum of 2 prime numbers?

   (a) 3

   (b) 5

   (c) 9 ← CORRECT

   (d) 10

Solution: 2 cannot be expressed as the sum of 2 primes, since 0 and 1 aren't primes. $4 = 2+2$, $6 = 3+3$, $8 = 5+3$, $10 = 5+5$, $12 = 5+7$, $14 = 7+7$, $16 = 11+5$, $18 = 11+7$, and $20 = 13+7$. This is also known as the goldbach's conjecture.

4. How many ways are there to arrange 4 identical people in a circle of 6 chairs such that rotations and reflections are considered the same?

   (a) 3 ← CORRECT

   (b) 6

   (c) 12

   (d) 24

   Solution: The two empty chairs can either be 0 chairs apart, 1 chair apart, or 2 chairs apart, resulting in $\boxed{3}$ possible arragements.

5. Let LCM $(x, y) = 546$. If the greatest prime factor of x is 7, what is the greatest prime factor of y?

   (a) 2

   (b) 3

   (c) 7

   (d) 13 ← CORRECT

   Solution: $546 = 2*3*7*13$. Since x doesn't contribute a 13 in the prime factorization, y must contribute it, so the answer is $\boxed{13}$.

6. Circle P and circle Q are concentric circles both centered at O. They have radii of 1 and 3, respectively. If chord AB lies in Q and is externally tangent to P, what is the length of AB?

   (a) $2\sqrt{3}$

   (b) $4\sqrt{2}$ ← CORRECT

   (c) $\sqrt{6}$

   (d) $2\sqrt{2}$

   Let the point where AB is tangent to P be called M. M is the midpoint of AB. Since AM is tangent to P, $\angle AMO$ is a right angle. That makes $\triangle AMO$ a right triangle with hypotenuse $AO = 3$ and leg $OM = 1$. By the Pythagorean theorem, $AM = \sqrt{3^2 - 1^2} = 2\sqrt{2}$. Therefore, $AB = 2AM = \boxed{4\sqrt{2}}$.

7. Let Grace have two bottles of fruit punch: bottle A is 1/5 orange juice, and bottle B is 1/3 orange juice. If Grace wants to mix a certain amount of punch from these bottles to make a new fruit punch that is 1/4 orange juice, what fraction of the new punch is from bottle A?

   (a) 1/5
   (b) 5/8 ← CORRECT
   (c) 3/4
   (d) 4/9

   Solution: Let x be the fraction of punch from bottle A. We have

$$\frac{x}{5} + \frac{1-x}{3} = \frac{1}{4} \to \frac{5-2x}{15} = \frac{1}{4} \to 20 - 8x = 15 \to x = \boxed{\frac{5}{8}}.$$

8. There are 20 students taking a math test at the FSF high school. If the school has two classrooms that can each hold at most 20 students, how many ways are there to arrange the students into classrooms? (Assume the classrooms are distinguishable and the students are indistinguishable).

   (a) 18
   (b) 21 ← CORRECT
   (c) 27
   (d) 36

   Solution: Let the rooms be A, B and the number of students in each room A and B be denoted as (x, y) where x is number in A and y is number in B. The possible combinations are $(5,0), (4,1), (3,2), (2,3), (1,4).(0.5) \to \boxed{6}$.

9. Let $P(x)$ be a quadratic polynomial such that $P(0) = 3$, $P(1) = 5$, and $P(10) = 113$. What is $P(6)$?

   (a) 20
   (b) 32
   (c) 45 ← CORRECT
   (d) 48

   Solution: Let $P(x) = Ax^2 + Bx + C$.

$$P(0) = C = 3$$
$$P(1) = A + B + C = A + B + 3 = 5, \text{ so } A + B = 2$$
$$P(10) = 100A + 10B + C = 100A + 10B + 3 = 113, \text{ so } 100A + 10B = 110$$

Since $A + B = 2$, $10A + 10B = 20$. Subtracting from $100A + 10B = 110$, we get $90A = 90$, or $A = 1$, and therefore $B = 2 - 1 = 1$. $P(x) = x^2 + x + 3$, so $P(6) = 36 + 6 + 3 = \boxed{45}$.

10. Circle ω has perpendicular chords AB and CD, and AB intersects CD at point P. If $AP = 4$, $BP = 9$, and $CP = 6$, what is the radius of ω?

    (a) 6
    (b) 11/2
    (c) 13/2 ← CORRECT
    (d) 8

    Solution: By power of a point, $AP \cdot BP = CP \cdot DP$, so $6DP = 36$ or $DP = 6$. Since $CP = DP$ and $AB \perp CD$, AB is the diameter of ω. Therefore, the radius is $(4 + 9)/2 = \boxed{13/2}$.

11. Let n be an integer such that there are exactly 4 primes between n^2 and $n^2 + n + 1$, exclusive. What is the smallest possible value of n?

    (a) 10 ← CORRECT
    (b) 19
    (c) 20
    (d) 21

    Solution: Substituting in answer choices, you can see that 10^2 is 100 and $10^2+10+1 = 111$, and the only primes in between are $101, 103, 107, 109$, exactly 4, and 10 is the smallest answer choice.

12. What is the sum of all integers n such that $n(n + 1)$ is a perfect square?

    (a) 1
    (b) 0
    (c) -1 ← CORRECT
    (d) No such integer n exists

    Solution: The only integers that satisfy this are -1 and 0, since the product becomes 0, a perfect square. $-1 + 0 = \boxed{-1}$.

13. Triangle ABC is a right triangle such that $\angle B$ is a right angle and $\angle A = 30°$. Point D is on AB such that $\angle BCD = 45°$. The length of $CD = 5$. The area of ABC can be expressed as $\frac{a\sqrt{b}}{c}$. What is $a + b + c$?

    (a) 25
    (b) 32 ← CORRECT
    (c) 35
    (d) 40

    Solution: $\triangle BCD$ is a $45 - 45 - 90$ triangle and $\triangle ABC$ is a $30 - 60 - 90$ triangle. CD is the hypotenuse of $\triangle BCD$, so $BC = \frac{5\sqrt{2}}{2}$. Then, $AB = \frac{5\sqrt{6}}{2}$, and the area of ABC is $\frac{(AB)(BC)}{2} = \frac{25\sqrt{3}}{4}$, therefore the answer is $25 + 3 + 4 = \boxed{32}$.

14. An ant is on a pentagon with vertices labelled 1, 2, 3, 4, and 5, in order. The ant has equal probability to move forward 1 step (at point 1, moving back one step goes to point 5). What is the probability the ant reaches point 4 in three steps if it starts at point 1?

    (a) 1/8 ← CORRECT
    (b) 1/6
    (c) 1/4
    (d) 1/3

    Solution: In 3 steps the only possible way is to go from 1 to 2 to 3 to 4. There are no other possible ways because if you go back 2 and forward 1, you land on 5. Back 1 forward 2 gives 2, and back 3 gives 3. So its $\boxed{1/8}$.

15. Billy bakes a batch of cookies. They're arranged on a pan in the shape of a rectangle. Then, Carla comes and eats a cookie. She realizes she can arrange the cookies into a perfect square. However, she is hungry, so she eats another one. She notices that she can arrange the remaining cookies into 10 different rectangles with positive integer dimensions (flips/rotations are considered distinct). How many cookies did Billy bake?

    (a) 38
    (b) 44
    (c) 50 ← CORRECT
    (d) 65

    Solution: Looking at the answer choices, (a) and (b) can already be eliminated since subtracting one does not give you a square number. Then, we can see that subtracting 2 from 50 and 65 gives 48 and 63, but 63 only has 6 factors when 48 has 10. Therefore, the answer is $\boxed{50}$.

16. **Free Response Question.** Sarah has three dice: one with 4 sides numbered $1-4$, one with 6 sides numbered $1-6$, and one with 8 sides numbered $1-8$. What is the probability that the sum of the value shown on the three dice after Sarah tosses them is 5?

Solution: The only way that the sum is 5 is if the values are $1, 2, 2$ or $1, 1, 3$.

The probability of getting a $1, 2, 2$ is $3 \cdot \frac{1}{4} \cdot \frac{1}{6} \cdot \frac{1}{8} = \frac{1}{64}$, as the 1 can either appear on the 4, 6, or 8 sided dice.

Similarly, the probability of getting a $1, 1, 3$ is also $\frac{1}{64}$ since the 3 can appear on any of the three dice.

Therefore, the probability of getting a sum of 5 is $\frac{1}{64} + \frac{1}{64} = \boxed{\frac{1}{32}}$.

END OF TEST