FSF July Math Competition Middle School Exam

July 2024

1. What is the sum of the first 20 odd numbers?

   (a) 400

   (b) 410

   (c) 424

   (d) 438

   Solution: The sum of the first n odd numbers is equal to n^2. Therefore, the sum of the first 20 odd numbers is 20^2 or $\boxed{400}$.

2. What is the next number in the sequence with the first 5 terms $0, 1, 2, 5, 26$?

   (a) 59

   (b) 82

   (c) 127

   (d) 677

   Solution: The sequence can be represented by the equation $a_n = a_{n-1}^2 + 1$. Therefore, the next term in the sequence would be $26^2 + 1$ or $\boxed{677}$.

3. On a specific math test, the probability of you answering the first problem right is 1, the probability of answering the second problem right is $1/2$, third problem is $1/3$, and the nth problem is $1/n$. What is the probability you get all of the first 5 problems right?

   (a) $3/400$

   (b) $1/120$

   (c) $1/60$

   (d) $5/12$

Solution: The probability of getting all of the first five problems right would be the product of getting each individual question right. Therefore, it would be $1 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{5} = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \boxed{1/120}$.

4. A rectangular prism has a volume of 880, what is the smallest possible sum of its edges?

   (a) 96

   (b) 116

   (c) 136

   (d) 156

   Solution: The smallest possible sum of its edges would mean that they are as close to each other as possible. In this case, that would be 8, 10, and 11. Since this is a rectangular prism, each edge length appears 4 times (there are 12 edges total) so the the sum of its edges would be $4(8+10+11) = \boxed{116}$.

5. A palindrome is a number that reads the same left to right. How many 5 digit palindromes are there such that its digits are all odd or all even?

   (a) 125

   (b) 225

   (c) 250

   (d) 300

   Solution: If the digits are all odd, there are 5 options for the ten-thousands and ones digit (they're the same), 1, 3, 5, 7, and 9. The same applies for the thousands and tens digit, and also the hundreds digit. There are three digits total to pick, and there are 5 options for each digit, therefore there are $5 \cdot 5 \cdot 5 = 125$ palindromes with all odd digits. The same applies for even digits, where the possible digits can be 0, 2, 4, 6, and 8. However, the ten-thousands digit cannot be 0, therefore there are 4 options for the ten-thousands and ones digit and 5 options for the others. Therefore, there are $4 \cdot 5 \cdot 5 = 100$ palindromes with all even digits. Adding these up, there are a total of $\boxed{225}$ palindromes with all odd or all even digits.

6. I pick 2 points A and B on the circumference of a circle with diameter 12. What is the probability that $AB < 6$?

   (a) 1/6

   (b) 1/4

   (c) 1/3

   (d) 1/2

Solution: Since the diameter of the circle is 12, the radius is 6. In order for $AB < 6$, the measure of $\widehat{AB}$ must be less than or equal to 60°. For anywhere that we put A on the circle, you can place B up to 60° to the right and up to 60° to the left of A. This adds up to 120°, and since there are 360° total in a circle, the probability that $\widehat{AB} < 60°$ or $AB < 6$ is $\frac{120°}{360°}$, or $\boxed{1/3}$.

7. How many positive integers less than 1000 have exactly 5 factors?

   (a) 1

   (b) 3

   (c) 10

   (d) 60

   Solution: The only numbers with exactly 5 factors are primes to the 4th power. So, the only numbers less than 1000 that are prime to 4th power are $2^4, 3^4, 5^4$ so total is $\boxed{3}$.

8. How many ways are there for 5 people Alice, Bob, Cindy, David, and Emma to stand in a line such that David must stand in front of Emma?

   (a) 20

   (b) 24

   (c) 30

   (d) 60

   Solution: Note that the number of ways to arrange the 5 people in a line where David is in front of Emma is the same as the number of ways to arrange such that Emma is in front of David. Since the total of these 2 add up to the total number of ways to arrange the people in a line. The answer is just $\frac{5!}{2}$ or $\boxed{60}$.

9. I will roll a die until I get a prime number, what is the probability that after 6 rolls I will still need to roll a 7th time?

   (a) 1/32

   (b) 1/64

   (c) 1/128

   (d) 1/256

   Solution: Needing a 7th roll means that you rolled 6 composite numbers in the first 6 rolls. Ther are 3 composites on a die so the answer is $(\frac{1}{2})^6$ or $\boxed{1/64}$.

10. In a triangle with side lengths 5, 6, 7, an altitude is drawn to the side of length 6. The point where the altitude meets the side splits the side into two segments, one longer and one shorter. What is the ratio of the longer segment to the smaller segment?

    (a) 2 : 1

    (b) 3 : 1

    (c) 4 : 1

    (d) 5 : 1

    Solution: Let the length of the smaller segment be x, and the length of the altitude be y. Then, by using the Pythagorean theorem, we can set up two equations:

    $$x^2 + y^2 = 5^2$$
    $$(6-x)^2 + y^2 = 7^2$$

    Then, by subtracting the first equation from the second, we get

    $$(6-x)^2 - x^2 = 24$$
    $$36 - 12x + x^2 - x^2 = 24$$
    $$36 - 12x = 24$$
    $$x = 1$$

    Therefore, the length of the shorter segment is 1, and the length of the longer segment is $6-1=5$, and thus the ratio of the longer segment to the shorter segment would be $\boxed{5:1}$.

11. In a certain container of floss, the floss has a circular cross section with radius 0.1 mm. The floss is then tightly wrapped around itself to form a cylinder of radius 1 cm and width 1 cm. If you use 10 cm of floss every day, how many days will it take for you to finish this container of floss?

    (a) 10

    (b) 100

    (c) 1000

    (d) 10000

    Solution: Since the floss wrapped around itself results in a cylinder with radius 1 cm and width 1 cm, the total volume of all the floss would be π. Let the length of the floss be l. The total volume of the floss should be $\pi l r_{floss}^2$, where r_{floss} is the radius of the circular cross section of the floss. Since we already know that the volume of the floss is π, and $r_{floss} = 0.1$ mm or 0.01 cm, we can solve for l: $(0.01)^2\pi \cdot l = \pi$. From this equation we can get that $l = 10000$ cm. Since you use 10 cm of floss per day, the amount of days it would take to finish this container of floss would be $10000/10 = \boxed{1000}$ days.

12. How many 4 digit palindromes are divisible by 3?

    (a) 30

    (b) 60

    (c) 120

    (d) 180

    Solution: By division rules, we know that if the sum of the digits in a number is divisible by 3, then the number itself is also divisible by 3. In a four digit palindrome $\overline{abba}$, the sum of its digits is $a+b+b+a = 2(a+b)$. So, in order for $\overline{abba}$ to be divisible by 3, $2(a+b)$ must also be divisible by 3. That also means that $a+b$ must be divisible by 3. Knowing that a cannot be 0 because it is the first digit, we can pair the set $a \in 3,6,9$ with $b \in 0,3,6,9$, resulting in 12 pairs, or the set $a \in 1,4,7$ with $b \in 2,5,8$, resulting in 9 pairs, or the set $a \in 2,5,8$ with $b \in 1,4,7$, also resulting in 9 pairs. Since there are $12+9+9 = 30$ total pairs, there are $\boxed{30}$ total 4 digit palindromes divisible by 3.

13. In a rectangular sheet of paper $ABCD$, $AB = 6$, $BC = 8$. The paper is folded across the line segment formed by the midpoints of AB and BC such that point B lands on point E. What is the distance of point E to AB?

    (a) 12/5

    (b) 192/55

    (c) 72/25

    (d) 87/10

    Solution: Let the distance from E to AB be x. Now, let the altitude of E to BC be N and the midpoint of BC be M. Consider right triangle NBE and NME. They share leg EN, also $EM = 4$ because of reflection symmetry, and $MN = 4-x$, $EB = 2 \cdot \frac{12}{5} = \frac{24}{5}$ and $NB = x$. So, we have $4^2 - (4-x)^2 = (24/5)^2 - x^2$ which simplifies to $8x = 576/25$ so $x = \boxed{72/25}$.

14. Fibonacci numbers are numbers that are equal to the sum of the previous two numbers in the sequence. It starts like this: 1, 1, 2, 3, 5, 8, etc. The first time a multiple of 5 shows up in this sequence is on the 5th term, which is 5. The third time a multiple of 5 shows up in this sequence occurs on the nth term. What is the value of n?

    (a) 14

    (b) 15

    (c) 16

    (d) 17

Solution: A multiple of 5 occurs once for every 5 terms in the Fibonacci sequence. Therefore, the third time a multiple of 5 shows up in this sequence would be $5 \cdot 3 = \boxed{15}$.

15. Let ABC be a triangle with $AB = 12$, $BC = 5$, $AC = 13$. Let the altitude from B to AC intersect AC at D. Now, in triangle BDC, let the altitude from D to BC intersect BC at D_1, and in triangle BD_1D, let the altitude from D_1 to CD intersect CD at D_2 and so on. If you add up all the altitudes made ($BD + DD_1 + D_1D_2 + ...$) as the number of altitudes get larger and larger, what does this number approach?

    (a) 9/2

    (b) 11/2

    (c) 13/2

    (d) 15/2

    Solution: Note that in triangle ABC, $AB \cdot BC = AC \cdot BD$. Both represent 2 times area of triangle, as its base times height. Similarly, it is true for each of the other smaller right triangles. So, in ABC, $BD = 60/13$, in triangle $BDC, DD_1 = \frac{60}{13} \cdot \frac{5}{13}$ because it is similar triangles as AC and BC are corresponding sides in triangles ABC and BDC. If we continue this, we have an infinite geometric series with first term 60/13, common ratio 5/13. So the answer is $\frac{60}{13} \cdot \frac{1}{1-\frac{5}{13}} = \boxed{15/2}$.

END OF TEST