FSF April Math Competition Solutions

April 2024

1. What is $(2 \cdot (0+2)) \cdot 4$?

   (a) 12
   (b) 16
   (c) 8
   (d) 20

   Solution: We have

   $$(2 \cdot (0+2)) \cdot 4 = 2 \cdot 2 \cdot 4 = 16,$$

   so the answer is B.

2. I have the string of letters $MATHELETESMATHELETES...$, where it is an infinite string where the word "MATHELETES" is repeated over and over, what is the 244th letter in this sequence?

   (a) M
   (b) T
   (c) H
   (d) E

   Solution: Notice that in the word $MATHLETES$ there are 10 letters, so every 10 letter it repeats. 244/10 has a remainder of 4, therefore, we want the 4th letter of $MATHLETES$ which is H so the answer is C.

3. A circle is inscribed inside a square of side length 8, what is the area of the circle?

   (a) 16π
   (b) 32π
   (c) 64π
   (d) 256π

   Solution: Since the circle and the square are both symmetric figures, when inscribed, if you connect the 2 points on opposite sides of the square that is also on the circle, that is the diameter of the circle, and also the side length of the square. So the diamter is 8, radius therefore is 4, so area is

   $$4^2 \cdot \pi = 16\pi,$$

   so the answer is B.

4. Anna, Benjamin, and Chloe are sharing a pizza. Anna takes $\frac{1}{5}$ of the pizza, Benjamin takes $\frac{1}{3}$ of the remaining pizza, and Chloe takes some too. After all three take their pizza, there is still $\frac{1}{15}$ of the pizza remaining. What fraction of the whole pizza did Chloe take?

   (a) $\frac{2}{5}$
   (b) $\frac{7}{15}$
   (c) $\frac{2}{3}$
   (d) $\frac{8}{15}$

   Solution: After Anna takes $frac15$, there will be $1 - \frac{1}{5} = \frac{4}{5}$ of the pizza left, Benjamin takes $\frac{1}{3}$ of the remaining pizza so the leftover amount will be $\frac{4}{5} - \frac{1}{3} \cdot \frac{4}{5} = \frac{8}{15}$. After Chloe takes some pizza there is only $\frac{1}{15}$ of the pizza remaining, so Chloe took $\frac{8}{15} - \frac{1}{15} = \frac{7}{15}$ of the pizza. So the answer is B.

5. Leo has an urn of balls. When he takes them out in groups of three, he has two balls remaining. When he takes them out in groups of five, he has three balls remaining. When he takes them out in groups of seven, he has four balls remaining. If he has less than 100 balls, how many balls are in his urn?

   (a) 33
   (b) 43
   (c) 53
   (d) 63

   Solution 1: Note: You can solve this question by just trying all the answre choices and see which one has all 3 conditions satisfied, but here is a legitimate solution: We will solve this with algebra. Since when dividing by 3, the remainder is 2, the total must be in the form $3x + 2$ for some x. Now since it has a remainder of 3 when divided by 5, we must have $3x + 2 - 3 = 3x - 1$ must be a multiple of 5. $3x - 1$ will be a multiple of 5 when x has a remainder of 2 when divided by 5, so $x = 5a + 2$, therefore we can now express the total as $3(5a + 2) + 2 = 15a + 8$. Finally, it has a remainder of 4 when divided by 7, so $15a + 8 - 4 = 15a + 4$ must be a multiple of 7. This can only happen when a has a remainder of 3 when divided by 7. So, $a = 7b + 3$ for some b. Therefore, the total can now be expressed as $15a + 8 = 15(7b + 3) + 8 = 105b + 53$. We see whenever b is a non-negative integer, all 3 conditions will be satisfied like we proved. So, the answer is 53 as it is achieved when b is 0 and if b is 1, $105b + 53$ will be larger than 100. So the answer is C.

6. How many ways are there to arrange the letters of the phrase "I love math" ignoring spaces?

   (a) 362880
   (b) 40320
   (c) 5040
   (d) 720

   Solution: Since spaces are ignored and all the letters are distinct, its just arranging 9 different letters. For the first letter, there are 9 choices, second letter 8 choices, 3rd 7, and so on. So, the answer is $9 \cdot 8 \cdot 7 \cdots 2 \cdot 1 = 362880$. So the answer is A.

7. Evaluate:

$$\frac{1}{4} \cdot \frac{2}{5} \cdot \frac{3}{6} \cdot \frac{4}{7} \cdots \frac{20}{23}$$

   (a) $\frac{1}{1771}$
   (b) $\frac{1}{1521}$

(c) $\frac{1}{23}$

(d) $\frac{1}{4}$

Solution: Notice when multiplied together, there are the numbers 4 to 20 on both the numerator and the denominator, so it telescopes and it is left with $\frac{1 \cdot 2 \cdot 3}{21 \cdot 22 \cdot 23}$ which simplies to A.

8. If C, A, T, represent real numbers, what is $C + A + T$?

$$3C + 2A + T = 56$$

$$2C + 3A + T = 49$$

$$3C + 3A + 6T = 69$$

(a) 12.5

(b) 21.75

(c) 42

(d) 174

Solution: If you add the 3 equations, you get $8c + 8A + 8T = 174$, divide by sides by 8 and you get B.

9. If I have 2 apples, 3 pairs, and 4 oranges, how many ways can I arrange the 9 fruits in a circle if the same type of fruits are indistinguishable?

(a) 60

(b) 140

(c) 180

(d) 40320

Solution: The number of ways to arrange 9 things in a circle is $8! = 40320$. But since same are not distinguishable, we get the answer as $\frac{40320}{4! \cdot 3! \cdot 2!} = 140$. Answer is B.

10. There are 40 pairs of socks in a drawer. 10 of these pairs contain 2 red socks, 20 pairs contain one red sock and one blue sock, and 10 pairs contain 1 blue sock and 1 green sock. I draw a pair of socks at random, I see 1 blue sock, what is the probability that the other sock is also blue?

Solution. If the other sock is blue, that means that the pair of socks consists of 2 blue socks. There are no pairs of 2 blue socks in the drawer, making the probability $0 \rightarrow \mathbf{A}$.

11. 24 students are in a class learning how to make cookies. The recipe asks them to put butter and sugar into the bowl. All the boys and one sixth of the girls put in butter, and all of the girls and one sixth of the boys put in sugar. How many students have a missing ingredient?

(a) 12
(b) 16
(c) 18
(d) 20

Solution. Since all the boys put in butter but only 1/6 of the boys put in sugar, 5/6 of the boys have a missing ingredient. Since all the girls put in sugar but only 1/6 of the girls put in butter, 5/6 of the girls have a missing ingredient. Therefore, adding the boys and girls together, 5/6 of the total population have a missing ingredient, and the answer to the problem is $24 \cdot \frac{5}{6}$, or $20 \rightarrow$ **D**.

12. How many integers n are there from 1 to 30 such that $n(n+1)(n+2)$ is divisible by 9?

    (a) 0
    (b) 8
    (c) 9
    (d) 10

    Solution: Since at most one of $n, n+1, n+2$ is divisible by 3, we must have one of them divisible by 9. So, n is either $0, 1$, or 2 less than a multiple of 9, there are 9 numbers that satisfy. $7-9, 16-18, 25-27$.

13. In a magic square, the sum of every row, column, and diagonal is the same. In a 3 by 3 square, if the top left square is 5, the center square is 4, and the bottom right square is 10, what is the sum of the numbers in all 9 squares?

    (a) 50
    (b) 51
    (c) 54
    (d) 57

    Solution. Since the top left square, center square, and bottom right square make up a diagonal, their sum is equal to the sum of every row. This means that the sum of the values in each row is $5+4+10 = 19$, and since there are 3 rows total, the sum of the numbers in all 9 squares would be $3 \cdot 19 = 57 \rightarrow$ **D**.

14. What is the sum of the first 20 terms of an arithmetic sequence with 2nd term 6 and 9th term 55?

    (a) 1200
    (b) 1310

(c) 1520

(d) 1600

Solution. The common difference of this arithmetic sequence can be found by finding the difference of 55 and 6 and dividing it by $9 - 2 = 7$. Therefore, we know that the common difference is 7. The 20th term of this sequence can be found by adding $20 - 2 = 18$ times the common difference to the 2nd term. Therefore, the 20th term is $6 + 18 \cdot 7 = 132$. The first term can be found similarly to get -1. The sum of the first n terms in an arithmetic sequence can be modeled by $\frac{n(a_1+a_n)}{2}$ where a_i is the ith term of the sequence. Plugging in values, we get that the sum of the first 20 terms of the sequence is $\frac{20(-1+132)}{2} = 1310 \to \mathbf{B}$.

15. Three unit fractions sum to 1. However, when you take their individual reciprocals and add them up again, the sum becomes 10. What is the denominator of the product of these three fractions?

(a) 36

(b) 48

(c) 32

(d) 24

Solution. Since the sum of the denominators is 10, and they are all positive integers, not many possibilities are left, as the denominators cannot be large numbers. Through some guess and check, it is easy to find that $\frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$ and $2 + 4 + 4 = 10$. Therefore, $\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{32}$, and the denominator of $\frac{1}{32}$ is $32 \to \mathbf{C}$.

16. A solid shape A has a surface area 12 and volume 24. It is then diluted so that the new solid, A', has a surface area of 48. What is the volume of A'?

(a) 216

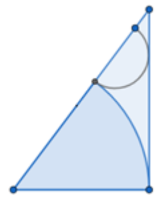
(b) 156

(c) 132

(d) 192

Solution. Surface area is measured in units^2, meaning that when it is diluted, the ratio of the surface areas is the square of the dilution factor. Since the ratio of the new surface area to the old surface area is $48/12 = 4$, the dilution factor is $\sqrt{4} = 2$. Volume is measured in units^3, so the ratio of the volumes is the cube of the dilution factor. Since the dilution factor is 2, the volume of A' is 2^3 or 8 times the volume of A. Therefore, the volume of A' is $24 \cdot 8 = 192 \to \mathbf{D}$.

17. In the figure below, a circle sector is drawn within a 3-4-5 right triangle. A semicircle is then drawn from the point where the circle sector meets the hypotenuse and is tangent to the leg of length 4. Find the radius of the semicircle.



(a) $\frac{1}{2}$
(b) $\frac{5}{8}$
(c) $\frac{3}{4}$
(d) 1

Solution. Let A, B, and C be the vertices of the triangle such that $AB = 3$, $BC = 4$, and $CA = 5$.Let D be the center of the semicircle, and let E be the point where the semicircle touches BC. Connect a line from D to E. Since BC is tangent to the semicircle at E, and D is the center of the semicircle, DE is perpendicular to BC and therefore $\triangle AED \sim \triangle ABC$. Using proportions from the similar triangles and letting r be the radius of the semicircle, $\frac{r}{3} = \frac{2-r}{5}$. Solving for r, we get $5r = 6 - 3r$, or $8r = 6$. Therefore, $r = \frac{6}{8} = \frac{3}{4} \to$ **C**.

18. The sum of the first two terms of a geometric sequence is 16. The product of the first three terms of that sequence is 1728. What is the fourth term?

(a) 108
(b) 256
(c) 324
(d) 728

Solution. Let a be the 1st term in the sequence and r be the ratio between each term. Then, we can write $a + ar = 16$ and $a \cdot ar \cdot ar^2 = (ar)^3 = 1728$. The second equation gives us $ar = 12$ after taking the cube root of both sides. Plugging this into the first equation, we get $a + 12 = 16$ or $a = 4$. Since $ar = 12$ and $a = 4$, $r = ar/a = 3$. The fourth term of the sequence can be expressed as ar^3, so the fourth term is $4(3)^3 = 108 \to$ **A**.

19. In a test consisting of two parts, six individuals received scores of 23, 24, 26, 28, 28, and 29 in the first part and 10, 10, 12, 12, 14, and 16 in the second part, though not in the same order. The overall winner is determined by the highest total score from both parts. What is the minimum possible total score the winner could have?

(a) 20

(b) 30

(c) 40

(d) 50

Solution. In order to find the minimum possible winning score, we need to have all the other scores as close to the winning score as possible. The average score (if all individuals received the same score) would be around 38.67 points, so ideally, the winning score should be just a little higher than this average. Looking at the answer choices, 40 is the closest value, so if the 12s and the 28s are matched, and the 24 and 26 are paired with 16 and 14, respectively, then all that is left is $23 + 10$ and $29 + 10$ which are both less than 40. Therefore, the answer would be $40 \to$ **C**.

20. If $\gcd(a, b) + \text{lcm}(a, b) + 135 = \gcd(a, b) * \text{lcm}(a, b)$, where a and b are positive integers. What is the sum of all possible values of $a + b$?

(a) 32

(b) 139

(c) 144

(d) 189

Solution: Note that $\gcd(a, b) \cdot lcm(a, b) = ab$, so we have

$$\gcd(a, b) + lcm(a, b) + 135 = ab.$$

Let d be the gcd, and $a = dx, b = dy$, so we have

$$d + dxy + 135 = d^2xy \to 135 = d(dxy - xy - 1).$$

Notice that d must be a factor of 135. If $d = 1$, then we get -1 on the right hand side which is impossible. If $d = 3$, we get

$$135 = 3(2xy - 1) \to \ 45 = 2xy - 1 \to xy = 23.$$

So, x, y are $23, 1$ in some order, as 23 is prime, doesn't matter which is which. So, a, b is $3, 69$ in some order, in this case $a + b$ is 72. Similarly, if you check $d = 5$, you get a, b is $35, 5$. And for $d = 9$ you get a, b is $18, 9$. So, the answer is $72 + 27 + 40 = 139$. Answer is B.